① a) $\mathbb{Q}$ je Lebegove mjere nula

b) $A \subseteq \mathbb{R}^m$ prebrojiv je Lebegove mjere nula

① a) $\mathbb{Q}$ prebrojiv

$\mathbb{Q} = \{q_1, \dots, q_m, \dots\}$, $\varepsilon > 0$.

$q_k \longrightarrow \left(q_k - \frac{1}{2}\frac{\varepsilon}{2^{k+1}}, q_k + \frac{1}{2}\frac{\varepsilon}{2^{k+1}}\right) = I_k$

$q_k \in I_k$, $\forall k$.

$\mathbb{Q} \subseteq \bigcup_{k=1}^{\infty} I_k$

$\sum_{k=1}^{\infty} \mu(I_k) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \frac{\varepsilon}{2} < \varepsilon$

b) Analogno.

② a) Neka je skup A Žordanove mjere nula.
Tada je A Lebegove mjere nula.

b) Obratno ne važi u opštem slučaju.
Ako je A kompaktan i Lebegove mjere nula
$\Rightarrow$ A Žordanove mjere nula.

③ a) Svaki interval I^m se može razbiti na prebrojivo mnogo disjunktnih podintervala t.d. $\{I_k^m\}_{k=1}^{\infty}$

$\sum_{k=1}^{\infty} \mu(I_k^m) = \mu(I^m)$

$A \subseteq \bigcup_{k=1}^{m} I_k$, $\sum_{k=1}^{m} \mu(I_k) < \varepsilon$

b). A kompaktan

$(\forall \varepsilon > 0)$ $(\exists \{I_k\}_{k=1}^{\infty})$ $A \subseteq \bigcup_{k=1}^{\infty} I_k$ $\sum_{k=1}^{\infty} \mu(I_k) < \varepsilon$

$\exists \{I_{k_1}, \dots, I_{k_m}\}$ t.d. $A \subseteq \bigcup_{j=1}^{m} I_{k_j}$

$\sum_{j=1}^{m} \mu(I_{k_j}) < \sum_{k=1}^{\infty} \mu(I_k) < \varepsilon$.

③ Unija najviše prebrojive familije skupova Lebegove mjere nula je skup Lebegove mjere nula.

(D) $\{A_k\}_{k=1}^{\infty}$ – skupovi L. mjere 0.

$$A_1 \longrightarrow \{I_{1,j}\}_{j=1}^{\infty} \qquad \sum_{j=1}^{\infty} \mu(I_{1,j}) < \frac{\varepsilon}{2^{1+1}}$$

$$A_k \longrightarrow \{I_{k,j}\}_{j=1}^{\infty} \qquad \sum_{j=1}^{\infty} \mu(I_{k,j}) < \frac{\varepsilon}{2^{k+1}}$$

$\vdots$

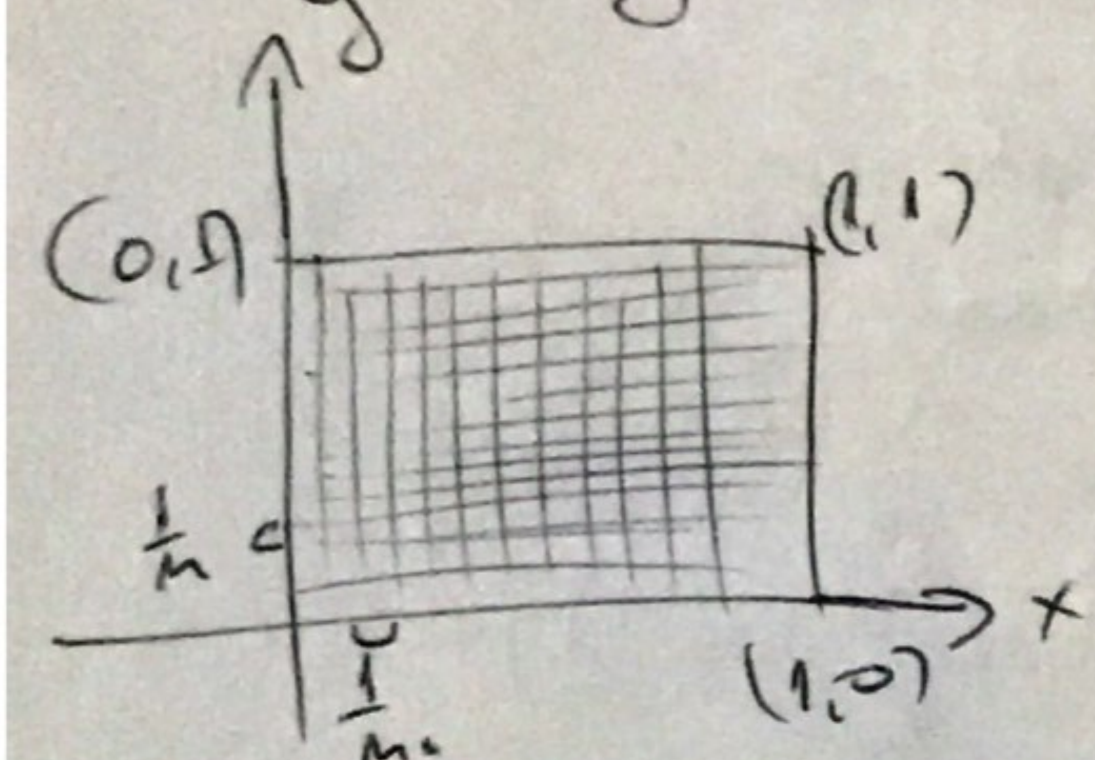
$$\bigcup_{j,k=1}^{\infty} I_{k,j} \supseteq A \quad \text{i} \quad \sum_{j,k=1}^{\infty} \mu(I_{k,j}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mu(I_{k,j})$$

$A = \bigcup A_k$

$$< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon$$

④ $\iint_{[0,1]^2} x^2y\,dx\,dy$ $\qquad f \in C([0,1]^2) \Rightarrow f \in R([0,1]^2)$

$f = x^2y$ $\qquad$ Skup tačaka prekida je $\emptyset$



$\boxed{\text{Lema } 5.4,\ 5.5}$

$S(f,P,T) = \sum f(t_i)\mu(D_i)$ integralna suma

$x_i = \frac{i}{n}$ $\quad i = \overline{1,n}$ $\quad D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $\quad t_i \in D_i$, D_i razbijanje skupa D.

$y_j = \frac{j}{n}$ $\quad j = \overline{1,n}$

$$\lim_{d(P)\to 0} S(f,P,T) = \int_D f$$

$n \to \infty \Rightarrow d(P) \to 0$

$$\underline{I}(f,P) \le S(f,P,T) \le \overline{I}(f,P)$$

jednog kvadratića

$\mu(D_{ij}) = \frac{1}{n^2}$ – uzimamo tačku $\left(\frac{i}{n}, \frac{j}{n}\right)$ (iz ovog kvadratića =)

$$\underbrace{\sum_{i,j=1}^{n} \frac{i^2}{n^2} \cdot \frac{j}{n} \cdot \frac{1}{n^2}}_{f(x_i, y_j)\,\mu(D_{ij})} = \frac{1}{n^5} \sum_{i,j} i^2 j = \frac{1}{n^5} \sum_{i=1}^{n} i^2 \sum_{j=1}^{n} j =$$

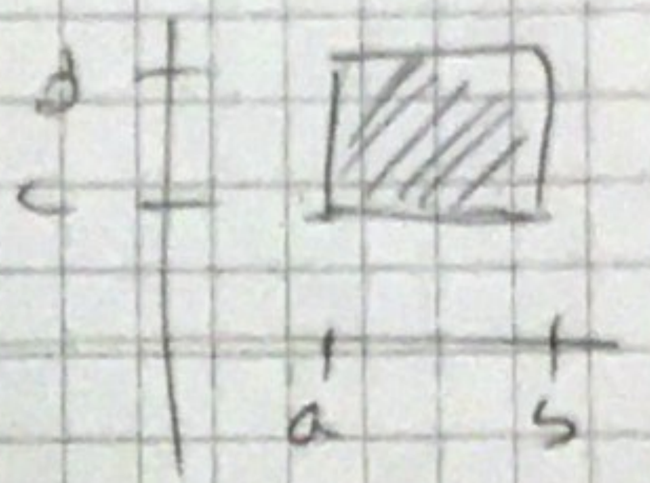
$$= \frac{1}{n^5} \cdot \frac{n(n+1)(2n+1)}{6} \cdot \frac{n(n+1)}{2} \xrightarrow{n \to \infty} \frac{2}{12} = \frac{1}{6}$$

Fubini

Teorema: Neka su X, Y intervali u $\mathbb{R}^m$ i $\mathbb{R}^n$. Ako je $f : X \times Y \subseteq \mathbb{R}^{m+n} \to \mathbb{R}$ integrabilna na $X \times Y$ tada je

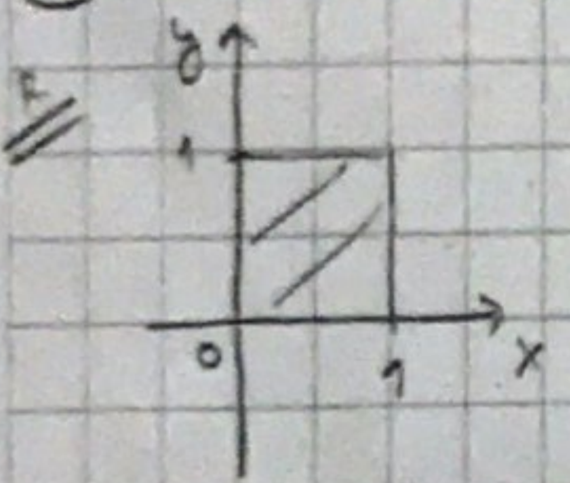
$$\iint_{X \times Y} f(x)\,dx\,dy = \int_X \left(\int_Y f(x,y)\,dy \right) dx$$

$$= \int_Y \left(\int_X f(x,y)\,dx \right) dy$$



$m = 1$
$n = 1$

(5.) Izračunati $\iint_D (x^2 + y^2)\,dx\,dy$ gdje je $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

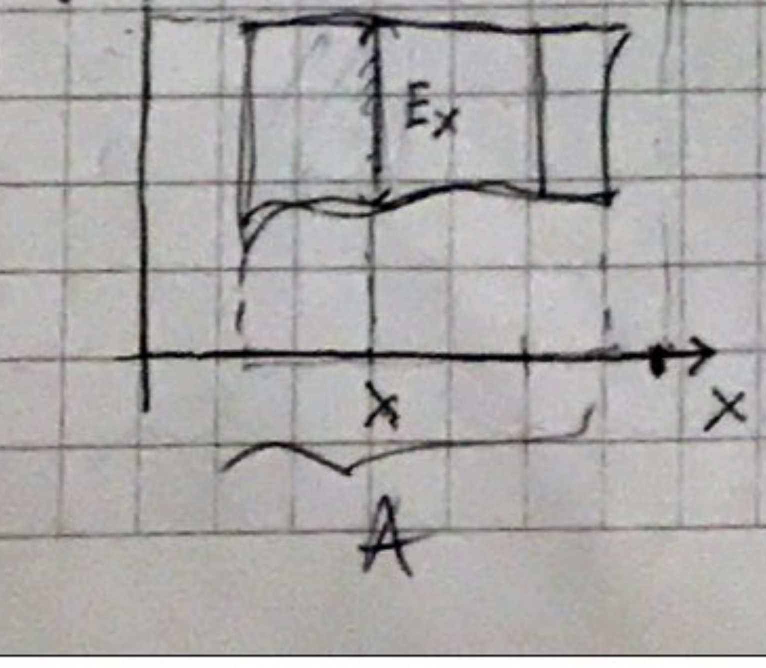


$$\iint_D (x^2 + y^2)\,dx\,dy = \int_0^1 \left(\int_0^1 (x^2 + y^2)\,dy \right) dx =$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \Big|_0^1 \right) dx = \int_0^1 \left(x^2 + \frac{1}{3} \right) dx =$$

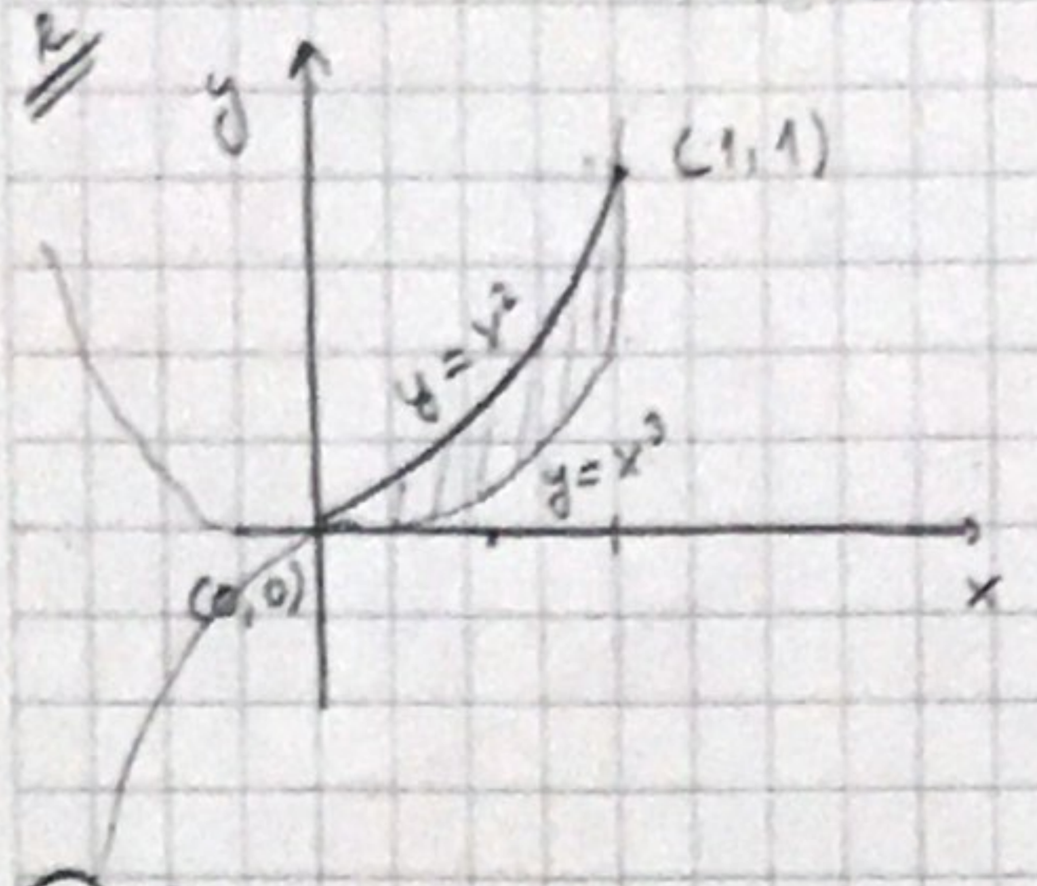
$$= \frac{x^3}{3} \Big|_0^1 + \frac{1}{3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Teorema: Neka je $E \subseteq \mathbb{R}^{m+n}$ mjerljiv skup i neka je A ~~mjerljiva~~ projekcija skupa E na $\mathbb{R}^m$ tj. A je skup svih $x \in \mathbb{R}^m$ za koje je skup $E_x = \{ y \in \mathbb{R}^n \mid (x,y) \in E \}$ neprazan. Tada je E_x mjerljiv skup za skoro svako $x \in A$ i važi $\int_E f = \int_A \left(\int_{E_x} f(x,y)\,dy \right) dx$.



⑥ Izračunati integral $\iint_G (1+xy^2)\,dx\,dy$ pri čemu je G oblast ograničena f-jama $y=x^2$ i $y=x^3$.

R:



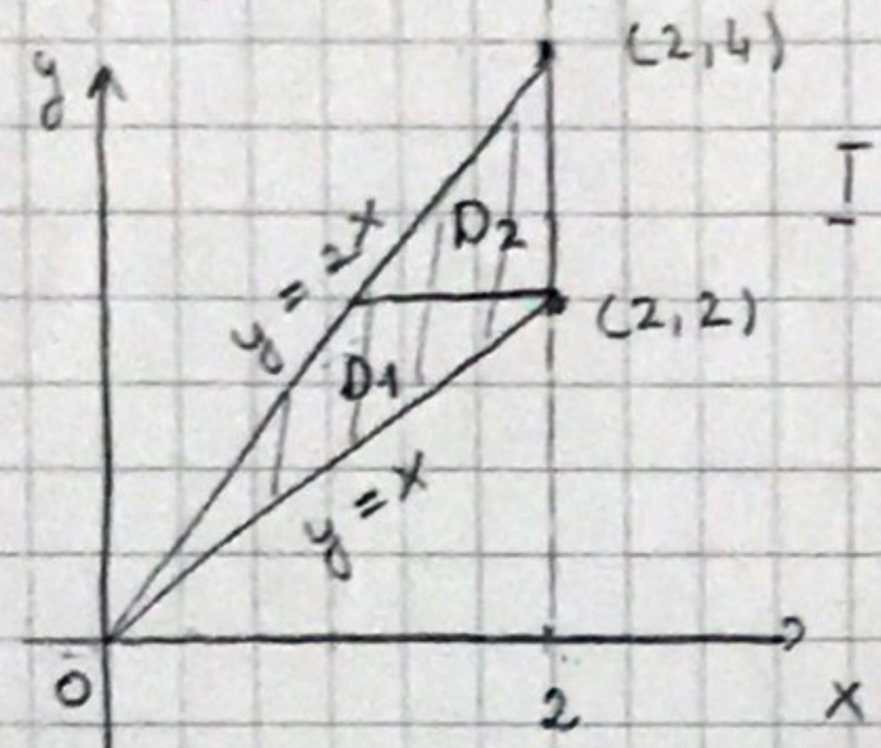
$$\iint_G (1+xy^2)\,dx\,dy = \int_0^1 \left(\int_{x^3}^{x^2} (1+xy^2)\,dy\right) dx =$$

$$= \int_0^1 \left[(x^2-x^3) + x\,\frac{y^3}{3}\Big|_{x^3}^{x^2} \right] dx = \dots$$

⑦ Promijeniti redoslijed integracije

a) $\int_0^2 dx \int_x^{2x} f(x,y)\,dy$

R:



$$I = \int_0^2 dy \int_{\frac{y}{2}}^{y} f(x,y)\,dx + \int_2^4 dy \int_{\frac{y}{2}}^{2} f(x,y)\,dx$$

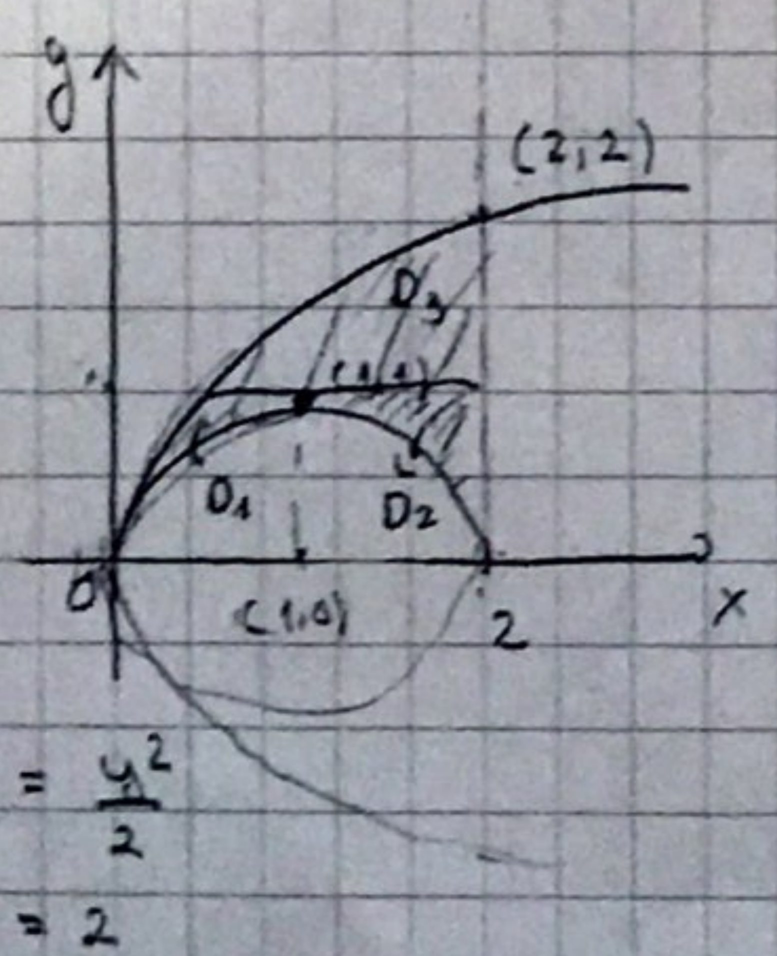
$y=2x \Rightarrow x=\frac{y}{2}$

$y=x$

$y=x$, $y=2x$

b) $\int_0^2 dx \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)\,dy$

R:



$y=\sqrt{2x} \quad /^2$

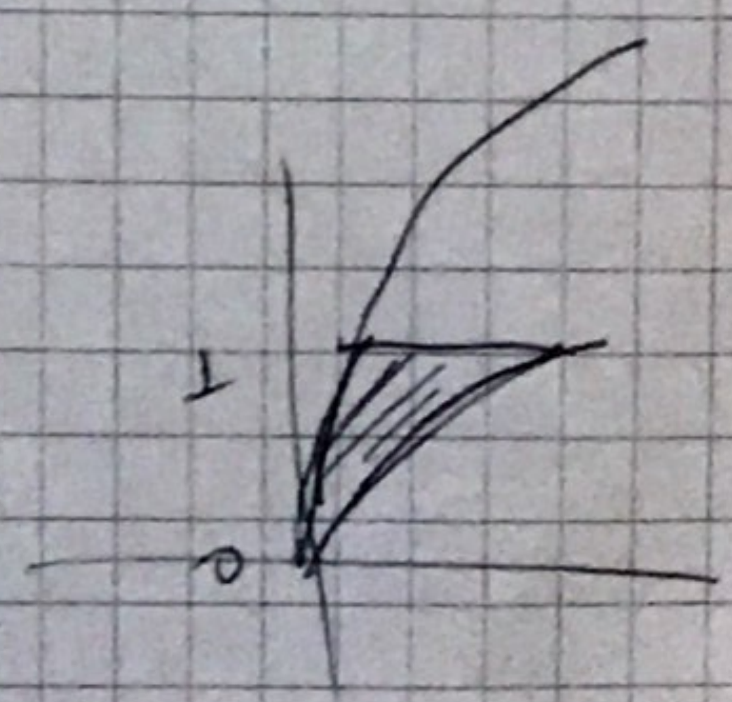
$y^2 = 2x \Rightarrow x = \frac{y^2}{2}$ parabola

$y=\sqrt{2x-x^2} \quad /^2$

$y^2 = 2x - x^2$

$x^2-2x+y^2+1 = 1$

$(x-1)^2+y^2 = 1$



$x = \frac{y^2}{2}$

$x = 2$

$(x-1)^2+y^2 = 1 \Rightarrow (x-1)^2 = 1-y^2$

$(x-1) = \pm\sqrt{1-y^2}$

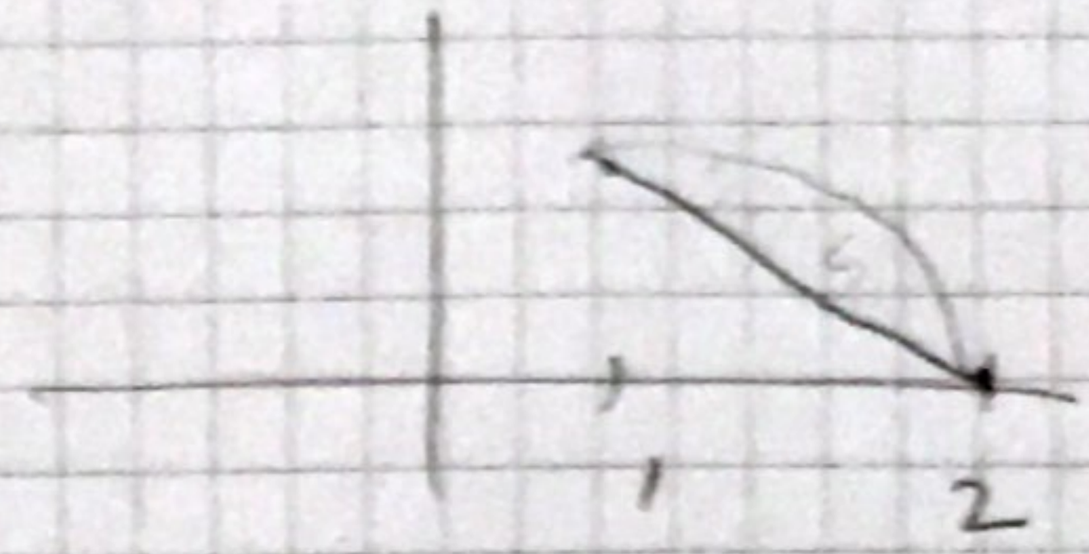
$x = 1 \pm \sqrt{1-y^2}$

- sa + desna polukr.
- sa − lijeva polukr.

$y \longrightarrow x$

$$\underbrace{\int_0^1 dy \int_{\frac{y^2}{2}}^{1-\sqrt{1-y^2}} f(x,y)\,dx}_{D_1} + \underbrace{\int_0^1 dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)\,dx}_{D_2} + \underbrace{\int_1^2 dy \int_{\frac{y^2}{2}}^{2} f(x,y)\,dx}_{D_3}$$

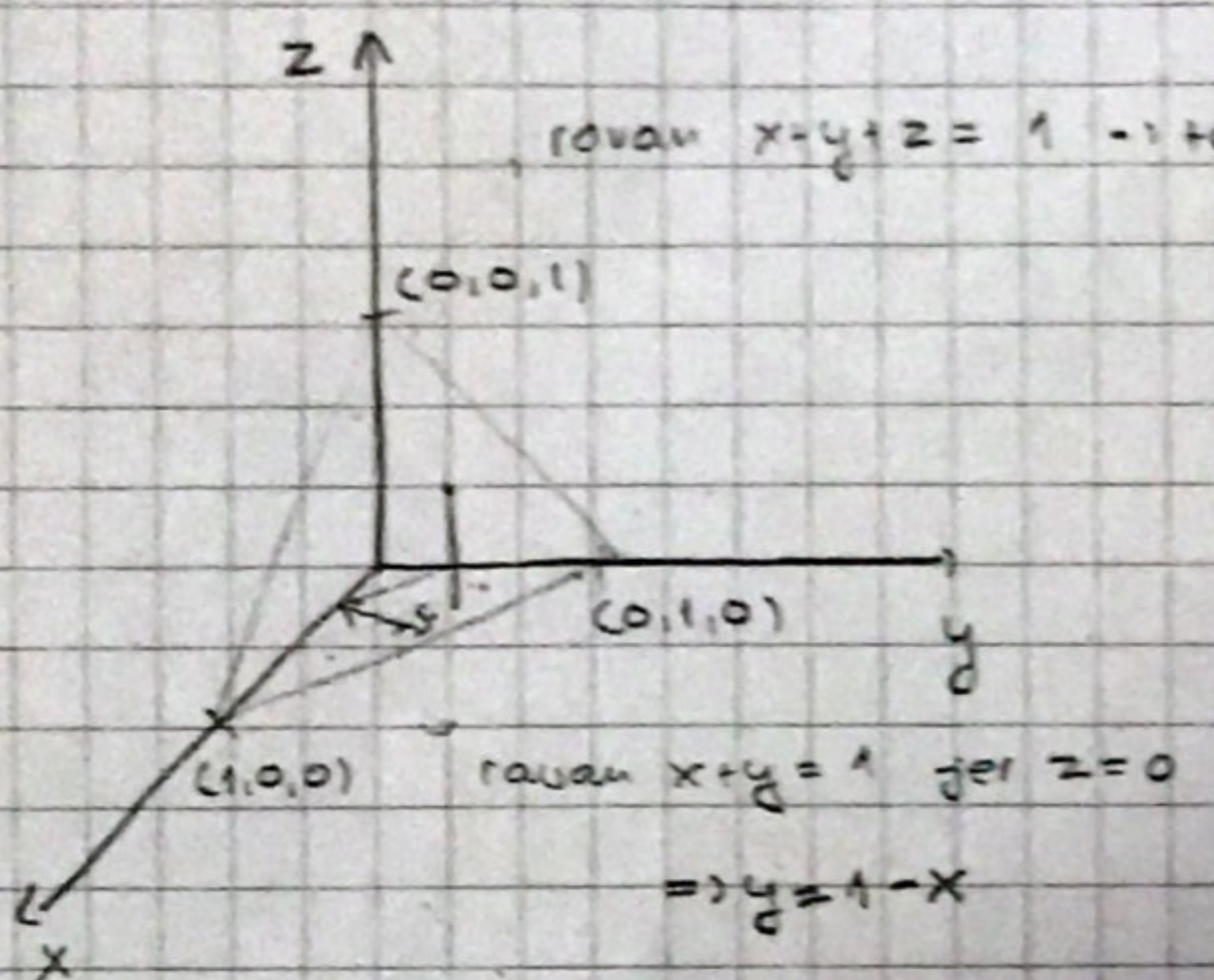
Za vježbu: $\displaystyle\int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy$

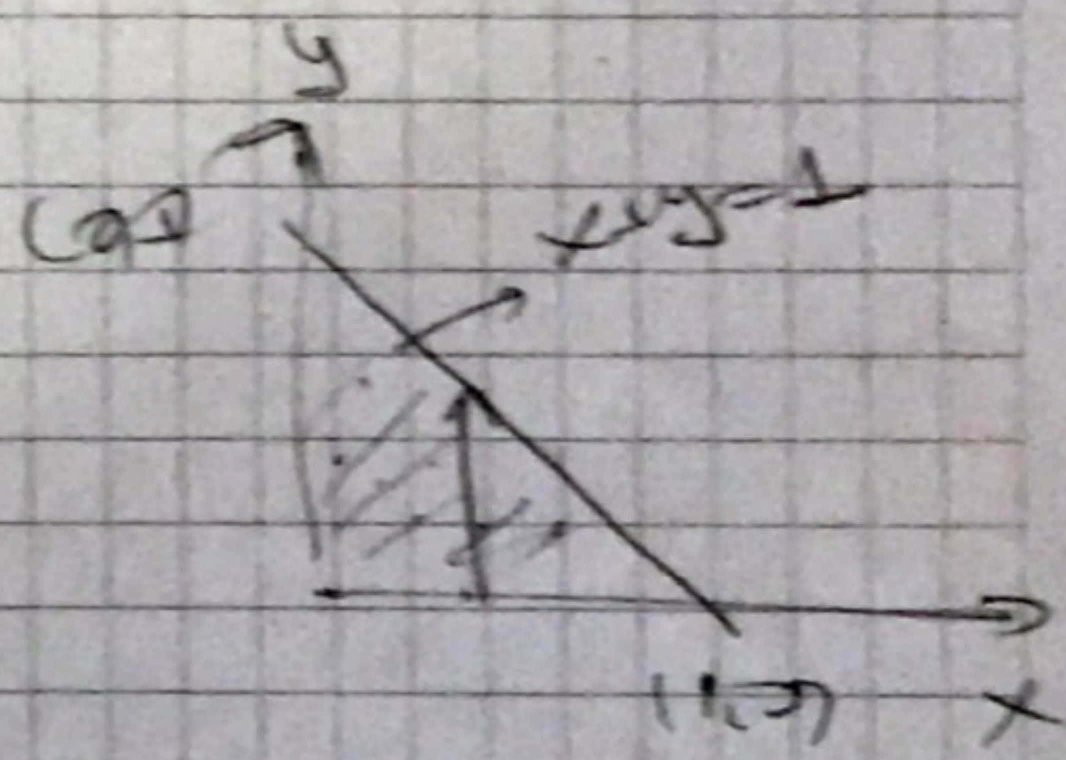


Reš: $\displaystyle\int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dy$

(8.) Izračunati int. po D $\displaystyle\iiint_D \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ D je oblast ograničena sa ravnima $x+y+z=1$, $x=0$, $y=0$, $z=0$. (uzima se pozitivan oktant)

R





$$\int_0^1 dx \int_0^{1-x} dy \underbrace{\int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3}}_{I_1} \qquad - t = 1+x+y+z$$

$$I_1 = \int_{1+x+y}^{2} \frac{dt}{t^3} = \frac{t^{-2}}{-2}\Bigg|_{1+x+y}^{2} = -\frac{1}{2}\left(\frac{1}{4} - \frac{1}{(1+x+y)^2}\right)$$

$$\int_0^{1-x} -\frac{1}{2}\left(\frac{1}{4} - \frac{1}{(1+x+y)^2}\right) dy$$

$\vdots$ $s = 1+x+y$

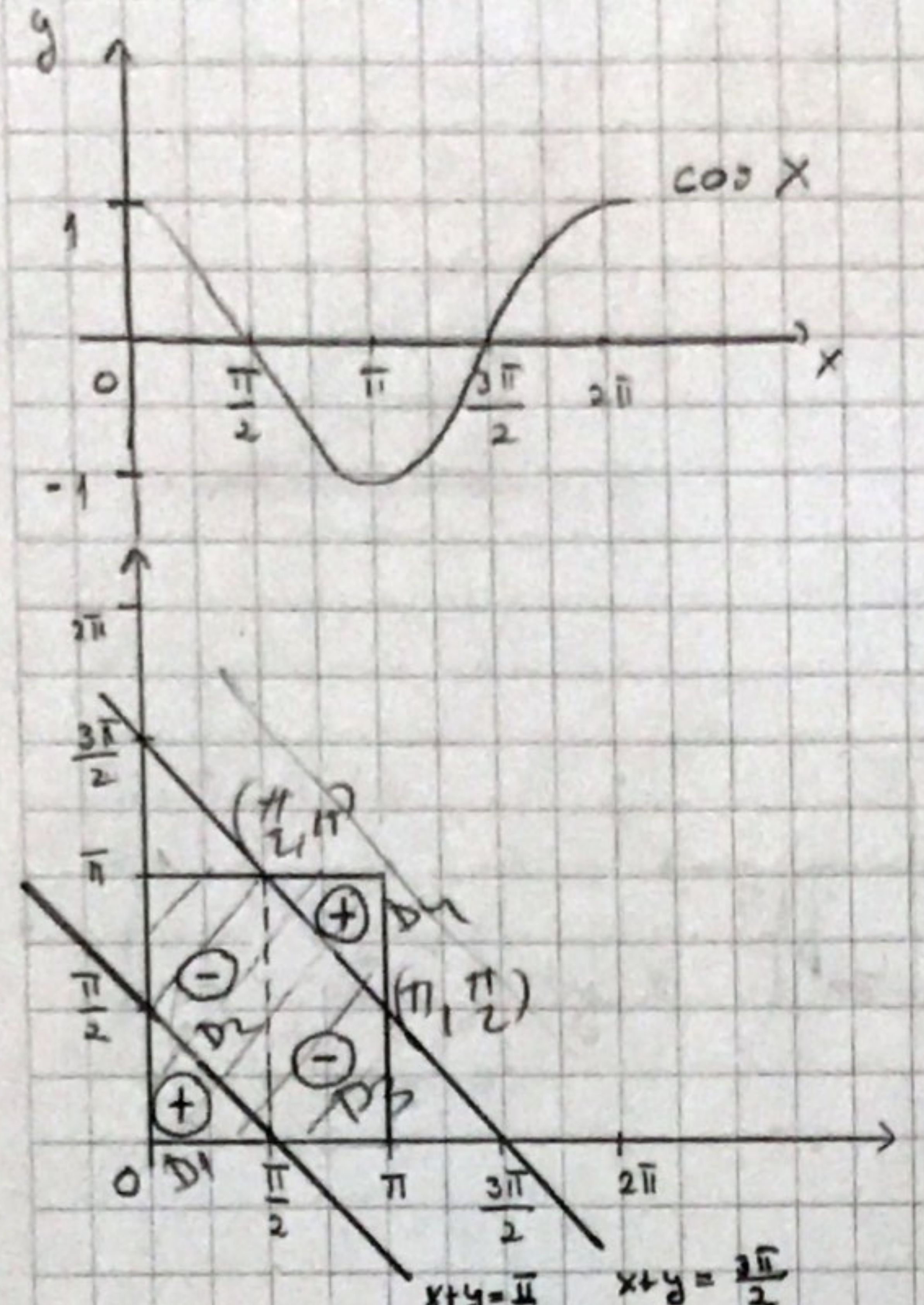
granice ...

$= -\frac{5}{6} + \frac{1}{2}\ln(2)$

9. Izračunati $\iint\limits_{[0,\pi]^2} |\cos(x+y)|\,dx\,dy$.

R//

$$|\cos(t)| = \begin{cases} \cos t, & t \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \\ -\cos t, & t \in (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$$



$x, y \in [0,\pi] \Rightarrow x+y \in [0, 2\pi]$

$t = x+y$

$0 < x+y < \frac{\pi}{2}$

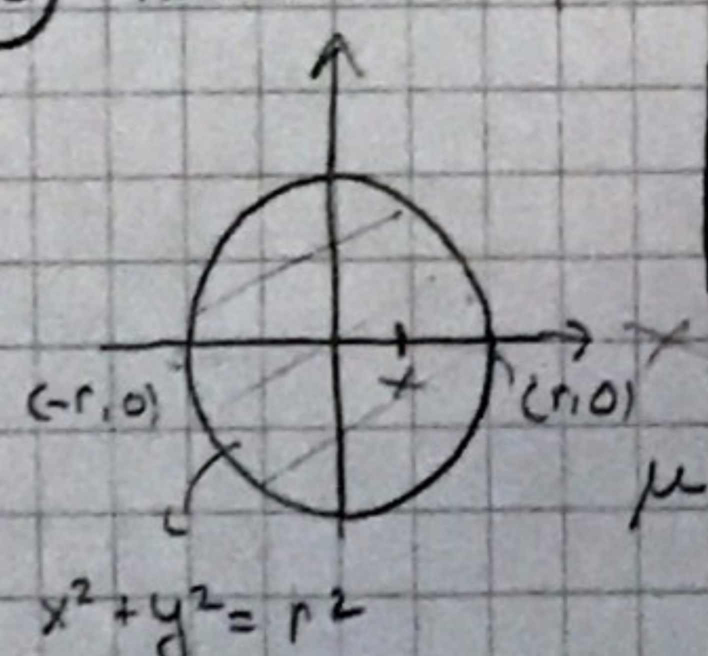
$\frac{\pi}{2} < x+y < \frac{3\pi}{2}$

$\frac{3\pi}{2} < x+y < 2\pi$

$$\iint\limits_{[0,\pi]^2} |\cos(x+y)|\,dx\,dy = \underbrace{\int_0^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}-x} \cos(x+y)\,dy}_{D_1} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} dx \int_{\frac{3\pi}{2}-x}^{\pi} \cos(x+y)\,dy}_{D_4}$$

$$- \underbrace{\int_0^{\frac{\pi}{2}} dx \int_{\frac{\pi}{2}-x}^{\pi} \cos(x+y)\,dy}_{D_2} - \underbrace{\int_{\frac{\pi}{2}}^{\pi} dx \int_0^{\frac{3\pi}{2}-x} \cos(x+y)\,dy}_{D_3}$$

10. Izračunati površinu kruga poluprečnika r.



$$\boxed{\mu(D) = \int_D 1}$$

← ako je $D \subset \mathbb{R}^n$ ograničen i mjerljiv po Žordanu, onda se njegova mjera ovako izražava

$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$|y| = \sqrt{r^2 - x^2}$

$-\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2}$

$$\mu(K) = \iint\limits_K 1\,dx\,dy = \int_{-r}^{r} dx \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 1\,dy =$$

$$= 2\int_{-r}^{r} \sqrt{r^2-x^2}\,dx = \begin{pmatrix} x = r\sin t \\ dx = r\cos t\,dt \end{pmatrix} =$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t}\; r\cos t\,dt =$$

$$2\cdot r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt = 2r^2\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt = 2r^2\frac{1}{2}\pi = \pi r^2$$

Posljedica

$\mu(E(a,b)) = \pi ab$

$\quad\Downarrow$

$\quad k$

Za vježbu: $x^2+y^2+z^2 \le R^2$ - mjera ~~zatvorenog~~ kupa (zapremina)

rešenje $= \dfrac{4\pi R^3}{3}$

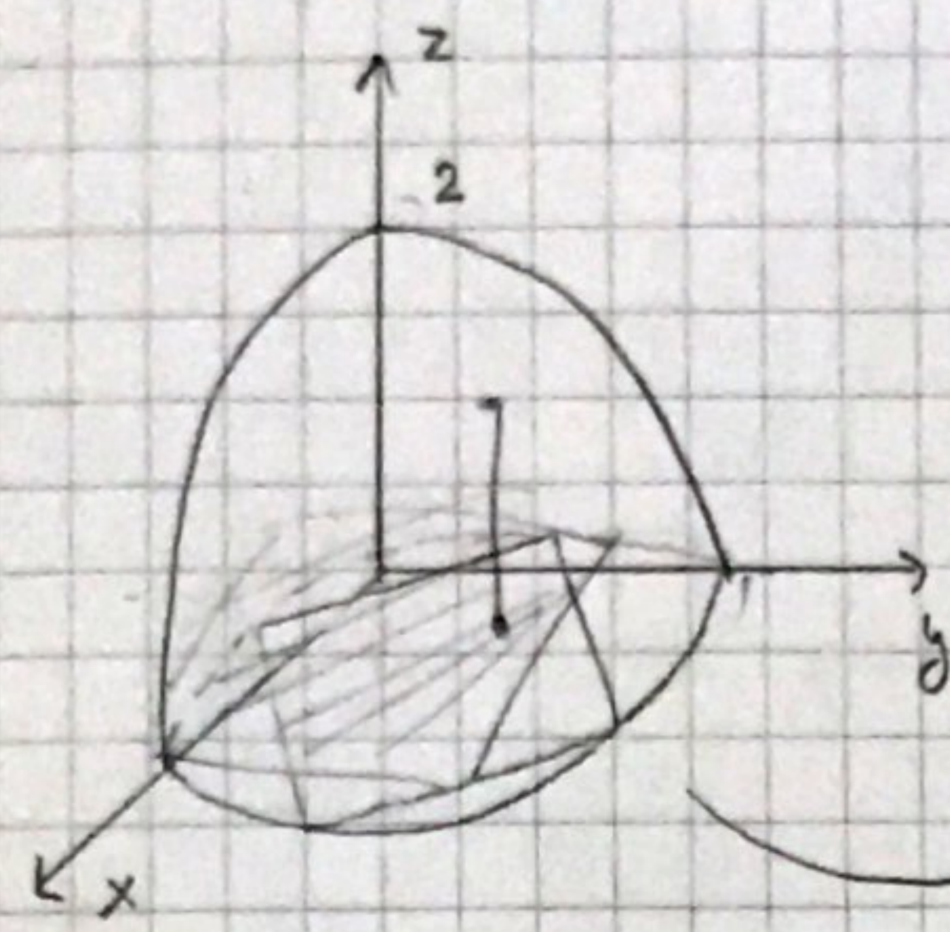
$$\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}dy\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}}1\,dz$$

$z = 2-x^2-y^2$

V

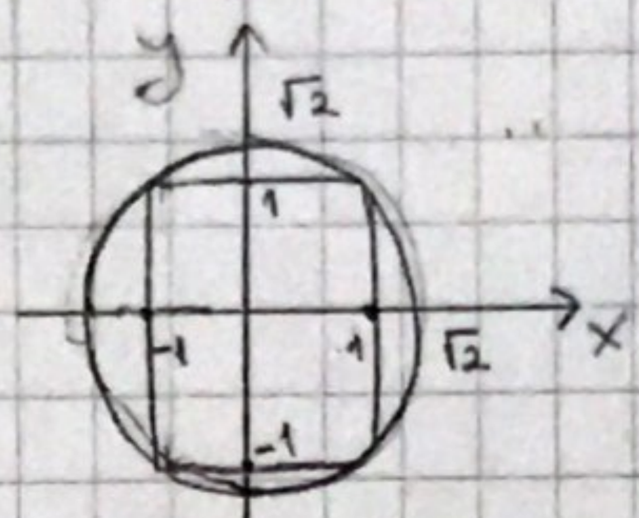
11. Izračunati zapreminu tijela ograničenog sa: $x=\pm 1$, $y=\pm 1$, $z=0$.

R



$$\begin{cases} z=0 \\ z = 2-x^2-y^2 \end{cases}$$

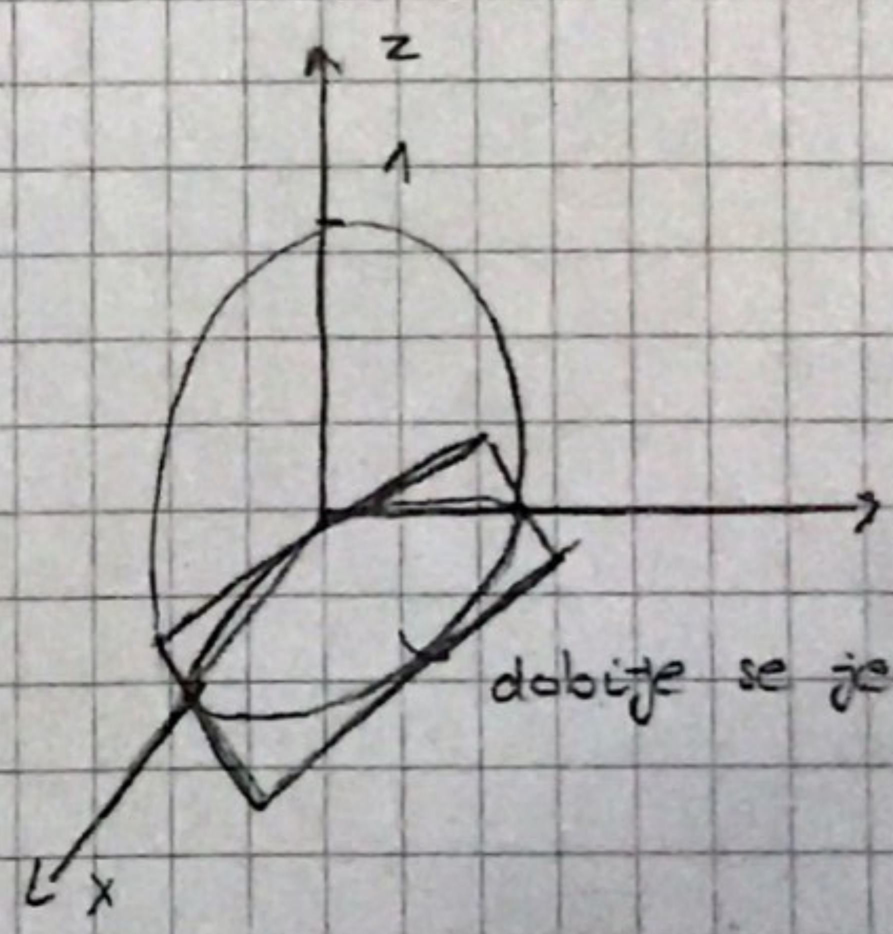
$x^2+y^2 = (\sqrt{2})^2$



$$\mu(V) = \iiint_V 1\,dx\,dy\,dz = \int_{-1}^{1}dx\int_{-1}^{1}dy\int_0^{2-x^2-y^2}dz$$

12. $\begin{cases} z = 1-x^2-y^2 \\ x=\pm 1 \\ y = \pm 1 \\ z=0 \end{cases}$

R



dobije se jedinična kružnica

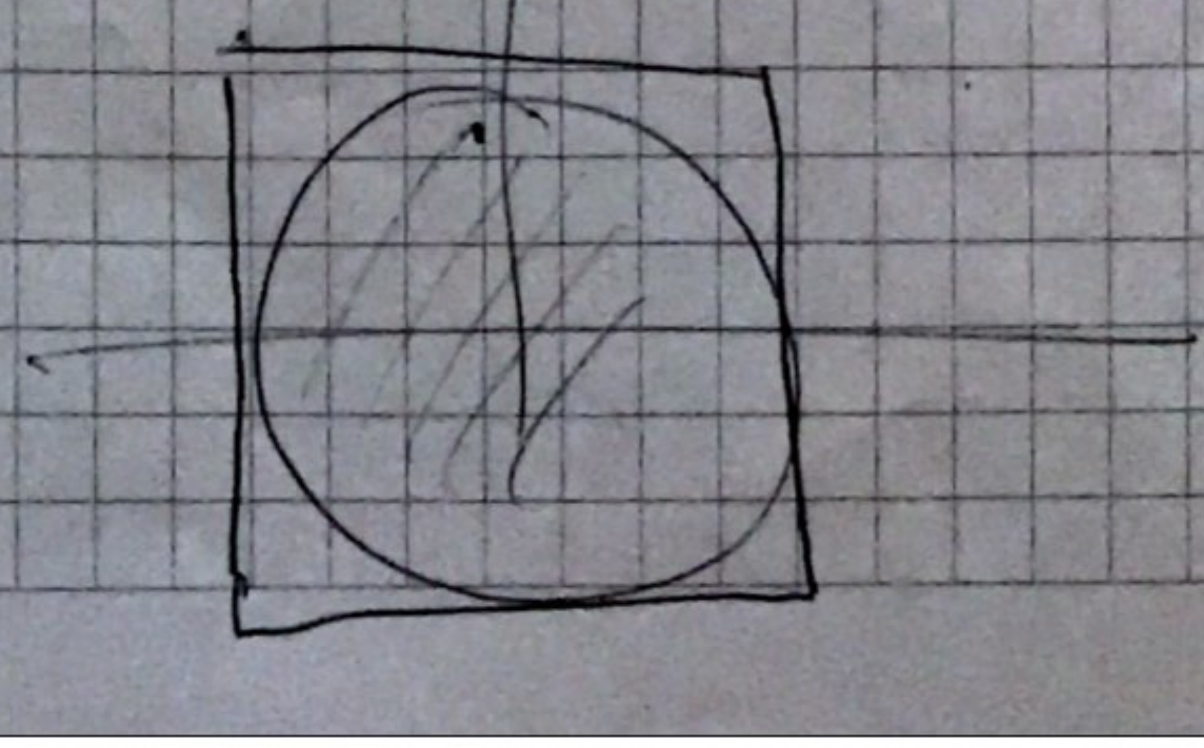
$$\begin{cases} z=0 \\ z = 1-x^2-y^2 \end{cases}$$

$\Rightarrow x^2+y^2=1$

a u isto vr. $x=\pm 1$, $y=\pm 1$

$$= \int_{-1}^{1}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_0^{1-x^2-y^2}dy$$

Geogebra 3d

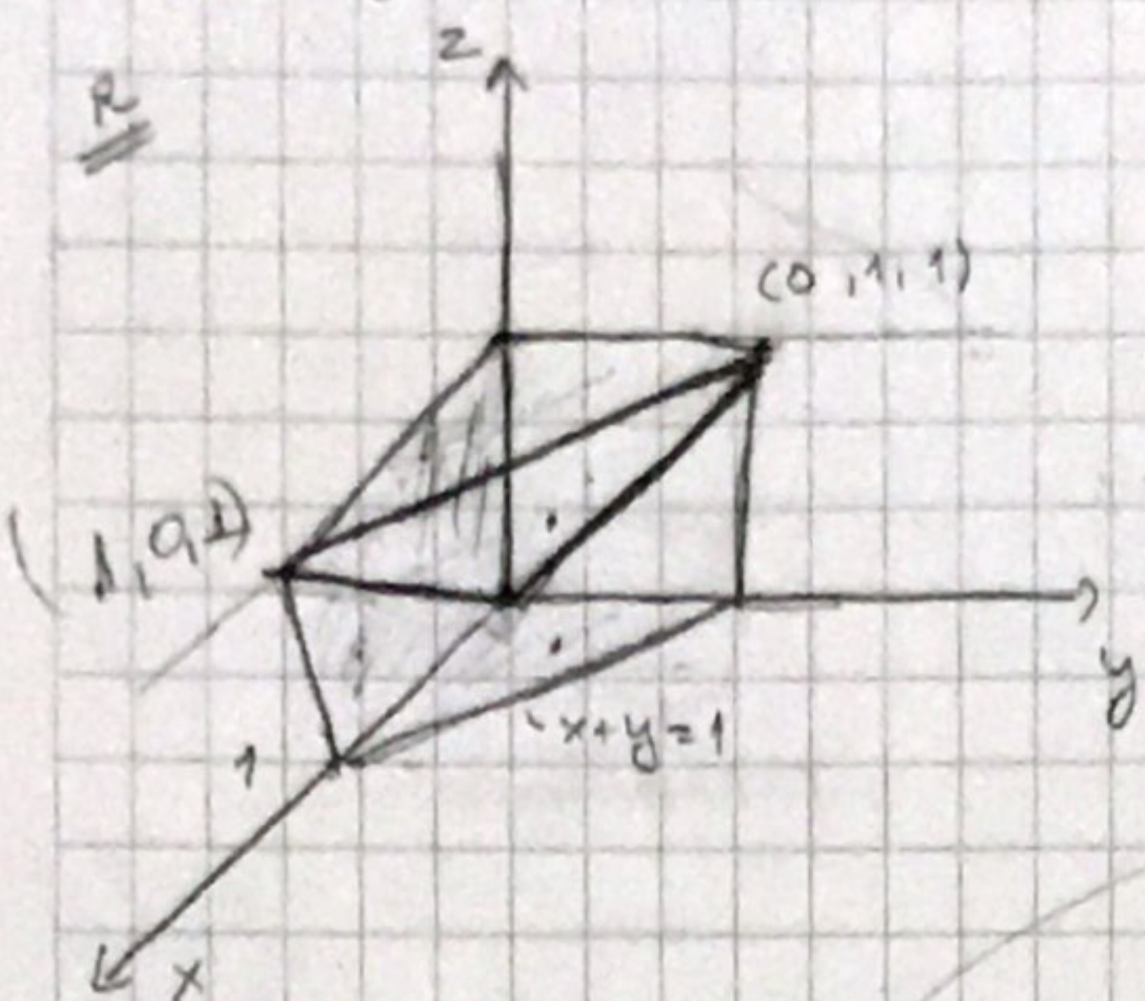


13. Promjeniti redoslijed integracije:

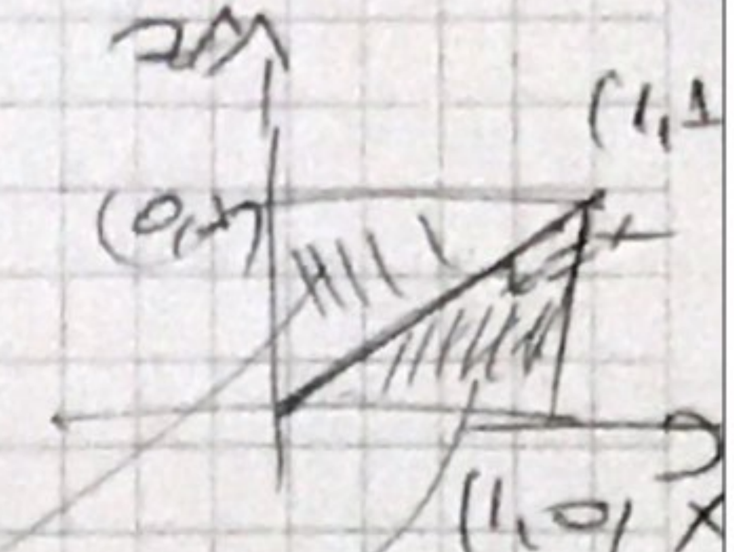
$$\int_0^1\int_0^{1-x}\int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$$

$z - x - y$

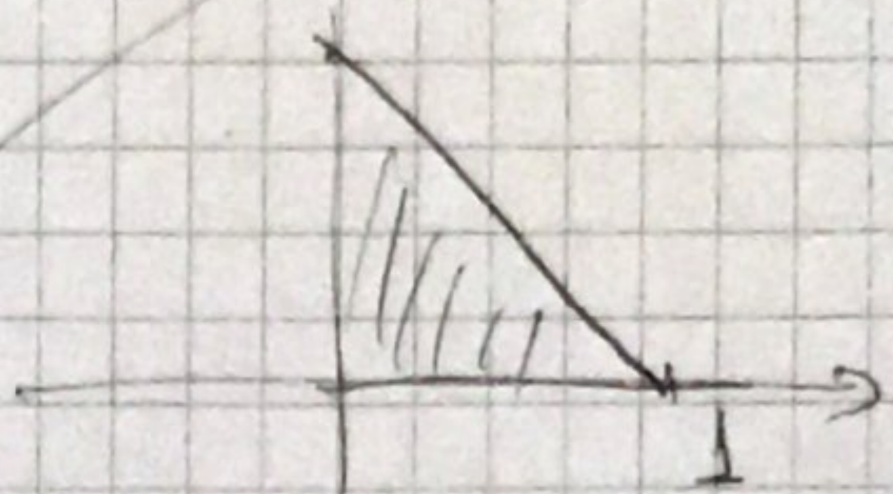
R:



$\begin{cases} z = x+y \\ x+y-z=0 \end{cases}$



$\left.\begin{cases} x+y = z \\ y = 0 \end{cases}\right\}\ x = z$



$z \in (0,1)$

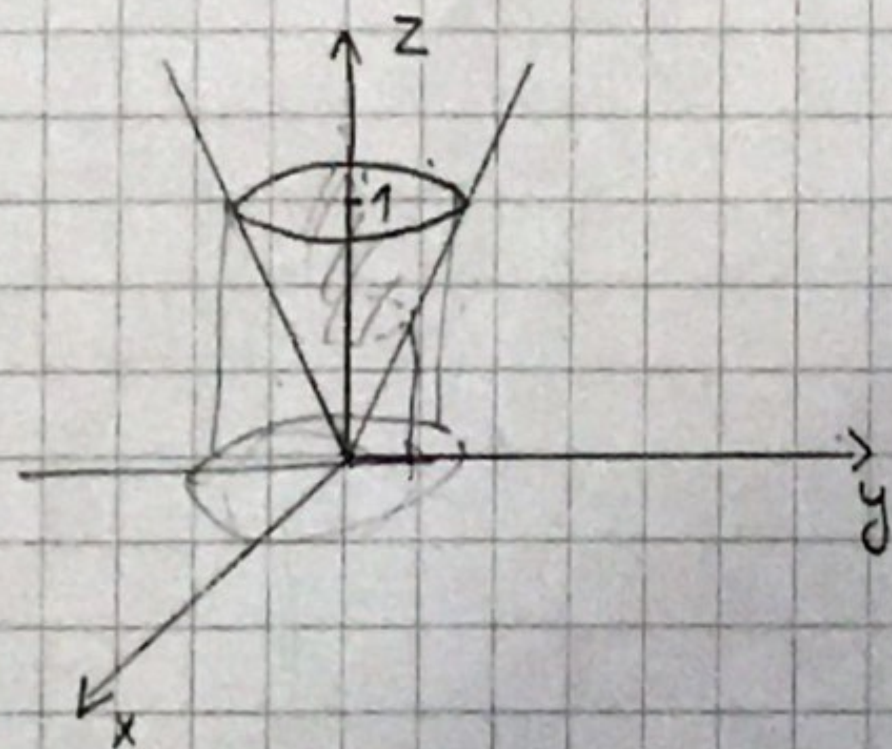
$$\int_0^1 dz \int_0^z dx \int_{z-x}^{1-x} f\,dy + \int_0^1 dz \int_z^1 dx \int_0^{1-x} f\,dy$$

14. Promjeniti redoslijed integracije.

$$\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}} f(x,y,z)\,dz$$

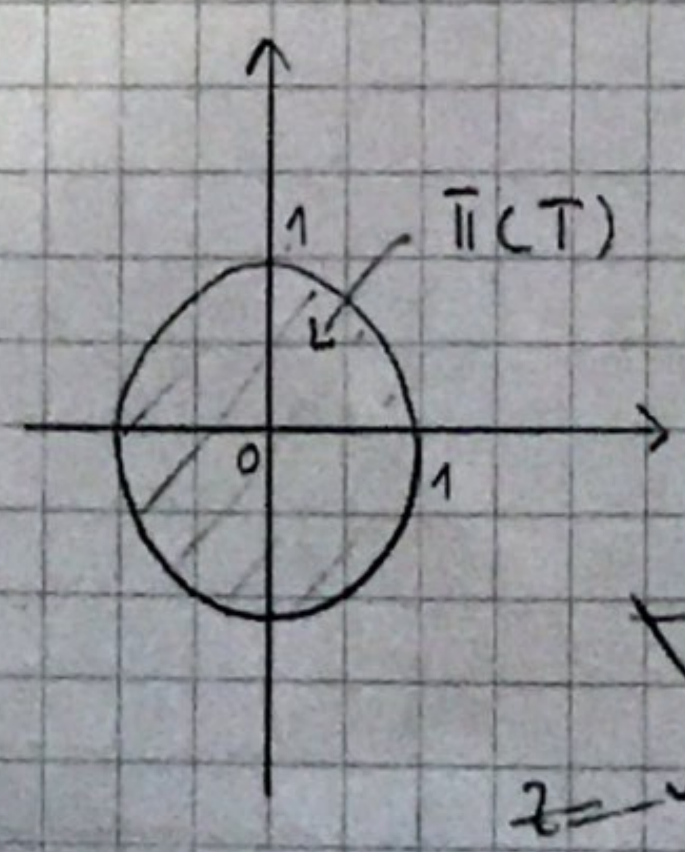
R:

$T = \left\{ (x,y,z) \in \mathbb{R}^3 \mid -1 \le x \le 1 \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\ ,\ \underbrace{\sqrt{x^2+y^2} \le z \le 1}_{\to\ 0 \le z \le 1} \right\}$



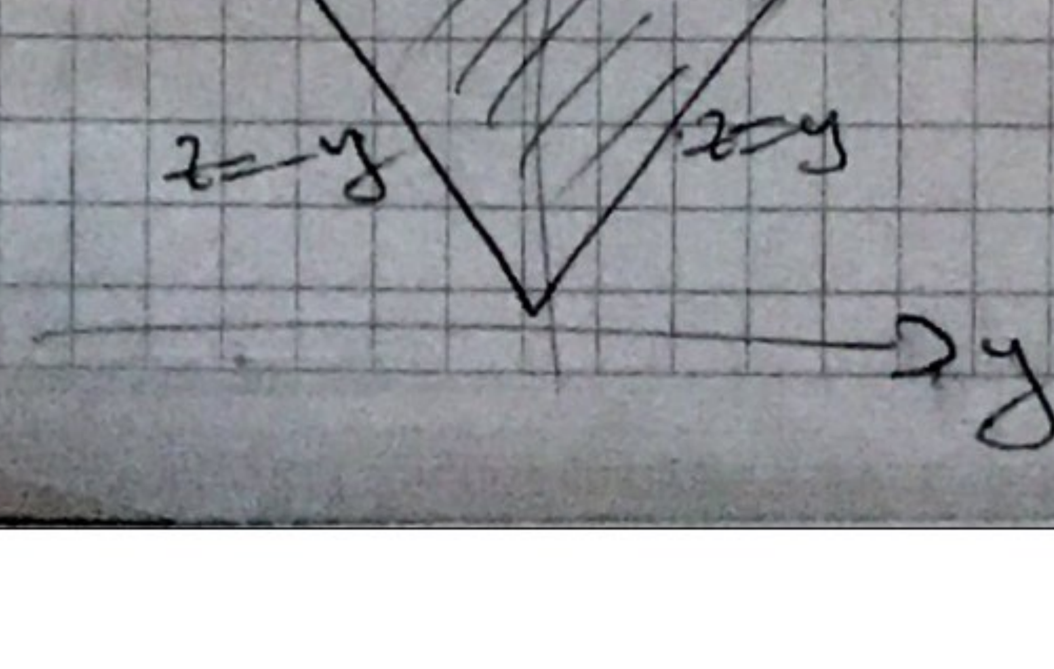
$z = 1$ ravan

$\sqrt{x^2+y^2} = z$ polovina konusa gdje je z pozitivno



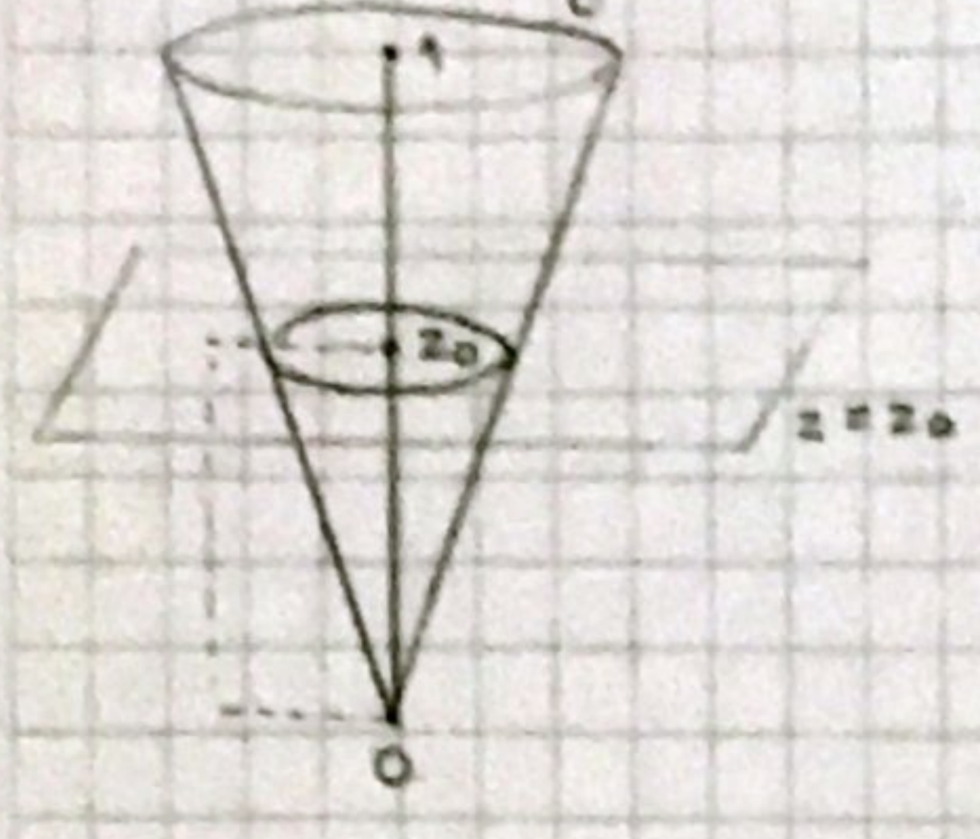
$\left.\begin{array}{l} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{array}\right\} \Leftrightarrow x^2+y^2 \le 1$

Projekcija tijela T na Oxy osu je krug $x^2+y^2 \le$



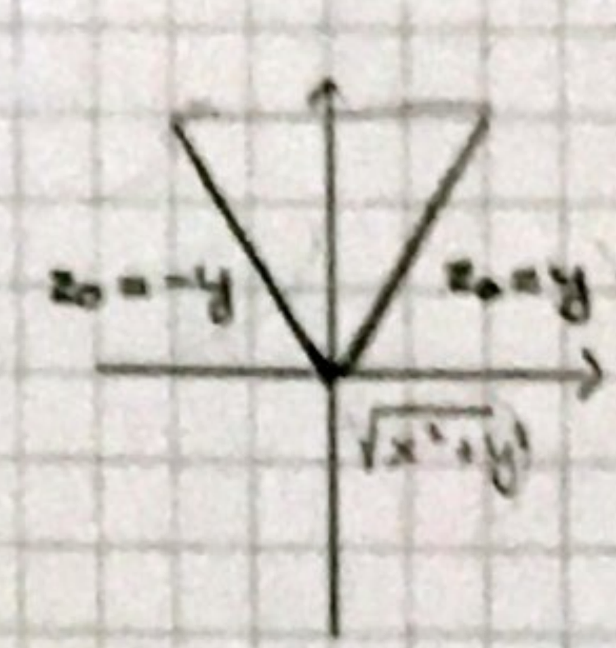
$\begin{cases} z = \sqrt{x^2+y^2} \\ x = 0 \end{cases}$

Ako fiksiramo vrijednost z_0, u presjeku ravni $z = z_0$ i tijela T dobija se krug poluprecnika z_0.



$\sqrt{x^2+y^2} \le z_0$

$x^2+y^2 \le z_0^2 \iff \begin{cases} -z_0 \le y \le z_0 \\ -\sqrt{z_0^2-y^2} \le x \le \sqrt{z_0^2-y^2} \end{cases}$



$$\Rightarrow J = \int_0^1 dz \int_{-z}^{z} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\, dx$$

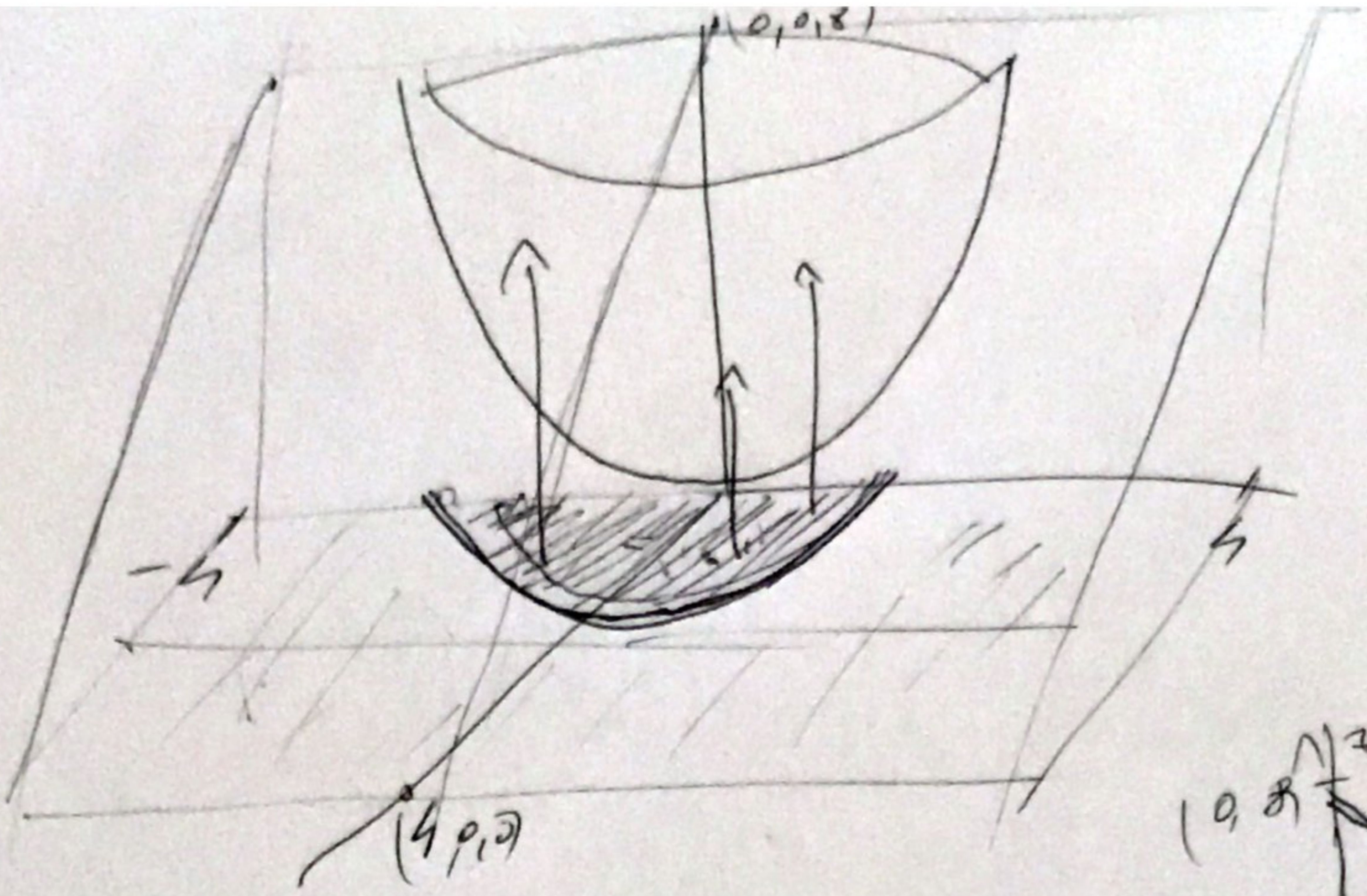
(15) Izračunati zapreminu skupa

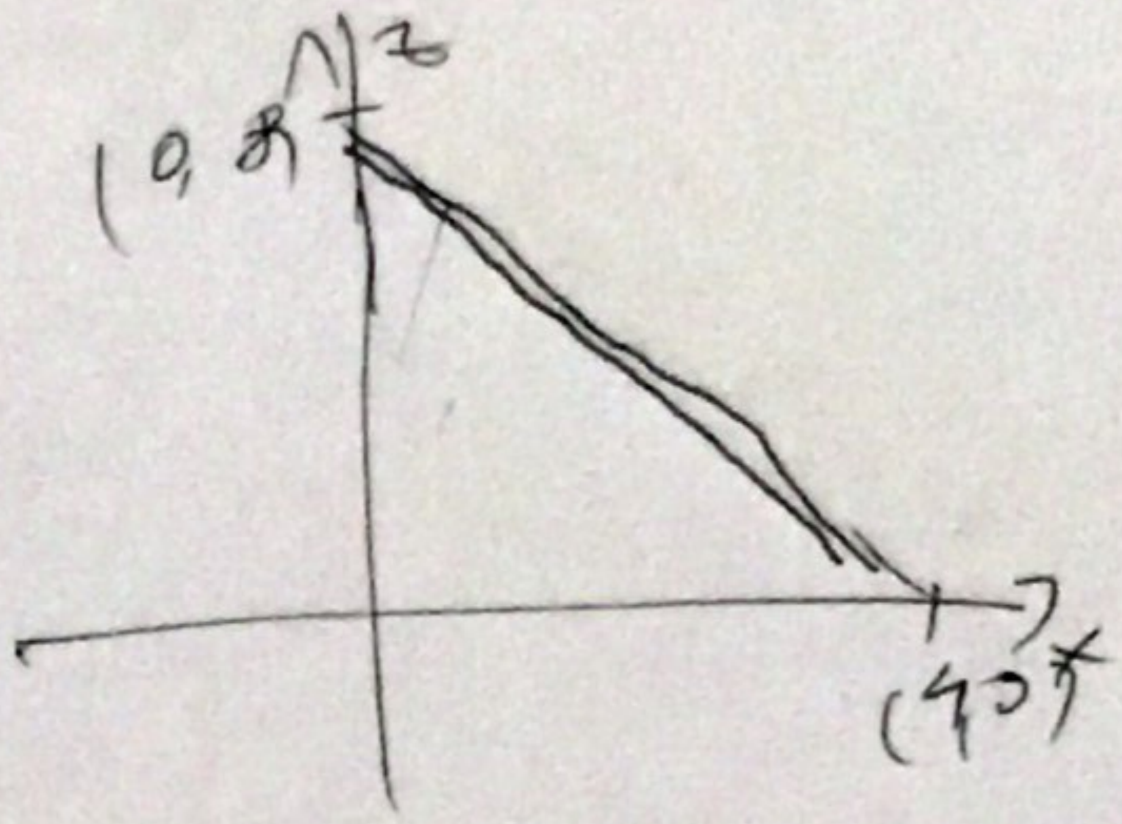
$z \ge 0,\ x \ge 0,\ -4 \le y \le 4,\ x^2 - y^2 \ge z,\ 2x + z \le 8.$

predstaviti preko int red. $x - y - z$

$$\int_0^2 dx \int_{-\sqrt{9-(x+1)^2}}^{\sqrt{9-(x+1)^2}} dy \int_0^{x^2-y^2} dz_{D_1} + \int_0^2 dx \int_{-4}^{-\sqrt{9-(x+1)^2}} dy \int_0^{8-2x} dz_{D_2}$$

$$+ \int_0^2 dx \int_{\sqrt{9-(x+1)^2}}^{4} dy \int_0^{8-2x} dz_{D_3} + \int_2^4 dx \int_{-4}^{4} dy \int_0^{8-2x} dz$$





$$\begin{cases} x^2+y^2=z \\ 2x+z=8 \end{cases} \Rightarrow \quad x^2+y^2 = 8-2x$$

$$(x+1)^2+y^2 = 9$$